

Fermi's "Golden Rule" and Non-Exponential Decay

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Abstract

A study is made of the behavior of unstable states in simple models which nevertheless are realistic representations of situations occurring in nature. It is demonstrated that a non-exponential decay pattern will ultimately dominate decay due to a lower limit to the energy. The survival rate approaches zero faster than the inverse square of the time when the time goes to infinity.

1 Introduction

In this article a study is made of the nature of the decay of unstable states in a nonrelativistic setting. One would expect this decay always to be exponential after some time, like the radioactive decay one finds in nature. It is the purpose of this article to demonstrate in a simple model with a ground state that next to a possible nondecaying bound state contribution there will be a non-exponential contribution to the decay pattern which will ultimately dominate the exponential decay. In practice this non-exponential contribution is extremely small and under normal conditions undetectable. Moreover it will arise only when one is able to keep a state coherent for a long time, which is not the case normally. This phenomenon is not unknown and is discussed in general terms in the literature¹, but a model demonstration may be helpful for its understanding.

Starting point is a simplified model giving rise to exact exponential decay. This will be worked out in Section 2. In Section 3 the model is modified as to make it more realistic. This modified model nevertheless leads to a superposition of exponential decay contributions of which one will ultimately dominate. Both models have an essential flaw which is caused by the fact that no absolute lower

¹J.J. Sakurai, "Modern Quantum Mechanics, Revised Edition", Supplement II (Addison Wesley, 1994); G-C. Cho, H. Kasari and Y. Yamaguchi, Prog. Theor. Phys. **90**, 803, (1993)

limit to the energy has been imposed. If an energy lower limit is taken into account then a non-exponential term appears to be unavoidable. Section 4 gives a discussion of this effect.

2 A simple, exactly solvable model

Consider a free Hamiltonian $H_0(= H_0^\dagger)$, a set of kets $\{|E\rangle\}$ and a particular ket $|a\rangle$ with the properties:

$$H_0|E\rangle = E|E\rangle, \quad H_0|a\rangle = \alpha|a\rangle, \quad -\infty < E < \infty \quad (1)$$

$$\langle E|E'\rangle = \frac{1}{\rho}\delta(E - E'), \quad \langle E|a\rangle = \langle a|E\rangle = 0, \quad \langle a|a\rangle = 1 \quad (2)$$

$$\rho \int_{-\infty}^{\infty} |E\rangle\langle E|dE + |a\rangle\langle a| = I, \quad \rho > 0 \quad (3)$$

Here ρ is the density of free energy-eigenstates, assumed to be independent of E . Furthermore, there is an interaction Hamiltonian $H'(= H'^\dagger)$ with the properties:

$$H'|E\rangle = a|a\rangle, \quad H'|a\rangle = \rho a \int_{-\infty}^{\infty} |E\rangle dE \quad (4)$$

Here we take a to be real, positive and independent of E . We have:

$$\langle a|H'|E\rangle = \langle E|H'|a\rangle = a \quad (5)$$

This is the transition matrix element.

Let the total Hamiltonian H be defined as:

$$H = H_0 + H' \quad (6)$$

We are interested in expressions of the form:

$$\langle a|e^{-\frac{it}{\hbar}H}|a\rangle \quad (7)$$

These are transition matrix elements for the transition of a state at time $t = 0$ to the same state at time $t \geq 0$. Traditionally, one reasons as follows: Up to time $t = 0$ the total Hamiltonian is H_0 which allows an experimenter to prepare a system in a pure energy eigenstate $|a\rangle$. Then, at time $t = 0$ the "perturbation" H' is "switched on" and at a later time t "switched off". From that time on one can carry out an analysis of the resulting state, in particular one can try to find out what the chances are that the original state is found back. According to Heisenbergs uncertainty principle for time and energy it takes an infinite amount of time to prepare the original state and to analyse the final results, but the available time to do so is indeed unlimited. The method of time dependent perturbation theory leads then to the famous "Golden Rule" of Fermi.²

²See any textbook on quantum mechanics.

In order to evaluate these matrix elements we put H in its spectral form:

$$H = \int_{-\infty}^{\infty} \lambda |\tilde{\lambda}\rangle \langle \tilde{\lambda}| d\lambda \quad (8)$$

$$H|\tilde{\lambda}\rangle = \lambda|\tilde{\lambda}\rangle \quad (9)$$

$$\langle \tilde{\lambda} | \tilde{\lambda}' \rangle = \delta(\lambda - \lambda') \quad (10)$$

$$\int_{-\infty}^{\infty} |\tilde{\lambda}\rangle \langle \tilde{\lambda}| d\lambda = I \quad (11)$$

Here we have made the assumption that H has no discrete eigenkets. We make the following expansion:

$$|\tilde{\lambda}\rangle = \int_{-\infty}^{\infty} f_{\lambda}(E) |E\rangle dE + c_{\lambda} |a\rangle \quad (12)$$

The normalization condition then gives:

$$\frac{1}{\rho} \int_{-\infty}^{\infty} f_{\lambda}^*(E) f_{\lambda'}(E) dE + c_{\lambda}^* c_{\lambda'} = \delta(\lambda - \lambda') \quad (13)$$

Next we solve the equation:

$$(H_0 + H' - \lambda) |\tilde{\lambda}\rangle = 0 \quad (14)$$

We have:

$$\begin{aligned} & (H_0 + H' - \lambda) |\tilde{\lambda}\rangle \\ &= (H_0 + H' - \lambda) \left[\int_{-\infty}^{\infty} f_{\lambda}(E) |E\rangle dE + c_{\lambda} |a\rangle \right] \\ &= \int_{-\infty}^{\infty} f_{\lambda}(E) [(E - \lambda) |E\rangle + a |a\rangle] dE \\ &\quad + c_{\lambda} [(\alpha - \lambda) |a\rangle + \rho a \int_{-\infty}^{\infty} |E\rangle dE] = 0 \end{aligned} \quad (15)$$

This can be satisfied only if:

$$(E - \lambda) f_{\lambda}(E) + \rho a c_{\lambda} = 0 \quad (16)$$

and

$$(\alpha - \lambda) c_{\lambda} + a \int_{-\infty}^{\infty} f_{\lambda}(E) dE = 0 \quad (17)$$

so that

$$f_{\lambda}(E) = -\rho a c_{\lambda} P \frac{1}{E - \lambda} + \beta \delta(E - \lambda) \quad (18)$$

and

$$\int_{-\infty}^{\infty} f_{\lambda}(E) dE = -\frac{(\alpha - \lambda) c_{\lambda}}{a} \quad (19)$$

From this we find:

$$-\rho a c_\lambda \int_{-\infty}^{\infty} P \frac{1}{E - \lambda} dE + \beta = -\frac{(\alpha - \lambda)c_\lambda}{a} \quad (20)$$

The integral is zero and we find:

$$\beta = -\frac{(\alpha - \lambda)c_\lambda}{a} \quad (21)$$

If we insert this we find the following expression for $f_\lambda(E)$:

$$f_\lambda(E) = -c_\lambda [\rho a P \frac{1}{E - \lambda} + \frac{\alpha - \lambda}{a} \delta(E - \lambda)] \quad (22)$$

Except for a phase factor the coefficient c_λ can be calculated from the normalization condition. We have:

$$\begin{aligned} & \int_{-\infty}^{\infty} f_\lambda^*(E) f_{\lambda'}(E) dE \\ &= c_\lambda^* c_{\lambda'} \int_{-\infty}^{\infty} [\rho a P \frac{1}{E - \lambda} + \frac{\alpha - \lambda}{a} \delta(E - \lambda)] \\ & \quad \cdot [\rho a P \frac{1}{E - \lambda'} + \frac{\alpha - \lambda'}{a} \delta(E - \lambda')] dE \\ &= c_\lambda^* c_{\lambda'} [\rho^2 a^2 \int_{-\infty}^{\infty} P \frac{1}{E - \lambda} \cdot P \frac{1}{E - \lambda'} dE \\ & \quad + \frac{\alpha - \lambda}{a} \int_{-\infty}^{\infty} \delta(E - \lambda) \cdot \rho a P \frac{1}{E - \lambda'} dE \\ & \quad + \frac{\alpha - \lambda'}{a} \int_{-\infty}^{\infty} \delta(E - \lambda') \cdot \rho a P \frac{1}{E - \lambda} dE \\ & \quad + \frac{(\alpha - \lambda)(\alpha - \lambda')}{a^2} \int_{-\infty}^{\infty} \delta(E - \lambda) \cdot \delta(E - \lambda') dE] \end{aligned} \quad (23)$$

The first integral on the right is not zero. We have:

$$\begin{aligned} & \int_{-\infty}^{\infty} P \frac{1}{E - \lambda} \cdot P \frac{1}{E - \lambda'} dE \\ &= \int_{-\infty}^{\infty} [\frac{1}{E - \lambda - i\epsilon} - i\pi \delta(E - \lambda)] \cdot [\frac{1}{E - \lambda' - i\epsilon} - i\pi \delta(E - \lambda')] dE \\ &= \int_{-\infty}^{\infty} \frac{1}{(E - \lambda - i\epsilon)(E - \lambda' - i\epsilon)} dE - i\pi \int_{-\infty}^{\infty} \delta(E - \lambda) \frac{dE}{E - \lambda' - i\epsilon} \\ & \quad - i\pi \int_{-\infty}^{\infty} \delta(E - \lambda') \frac{dE}{E - \lambda - i\epsilon} - \pi^2 \delta(\lambda - \lambda') \end{aligned} \quad (24)$$

Here the first integral is zero as follows from contour integration. The next two terms give:

$$-i\pi [\frac{1}{\lambda - \lambda' - i\epsilon} + \frac{1}{\lambda' - \lambda - i\epsilon}] = \frac{2\pi\epsilon}{(\lambda - \lambda')^2 + \epsilon^2} \quad (25)$$

We have:

$$\int_{-\infty}^{\infty} \frac{2\pi\epsilon}{(\lambda - \lambda')^2 + \epsilon^2} d\lambda = 2\pi \int_{-\infty}^{\infty} \frac{d\lambda}{1 + \lambda^2} = 2\pi^2 \quad (26)$$

Both terms together therefore give:

$$2\pi^2 \delta(\lambda - \lambda') \quad (27)$$

We therefore find:

$$\int_{-\infty}^{\infty} P \frac{1}{E - \lambda} \cdot P \frac{1}{E - \lambda'} dE = \pi^2 \delta(\lambda - \lambda') \quad (28)$$

The following two integrals together give:

$$\begin{aligned} & \frac{\alpha - \lambda}{a} \int_{-\infty}^{\infty} \delta(E - \lambda) \cdot \rho a P \frac{1}{E - \lambda'} dE \\ & + \frac{\alpha - \lambda'}{a} \int_{-\infty}^{\infty} \delta(E - \lambda') \cdot \rho a P \frac{1}{E - \lambda} dE \\ & = \rho(\alpha - \lambda) P \frac{1}{\lambda - \lambda'} + \rho(\alpha - \lambda') P \frac{1}{\lambda' - \lambda} = -\rho \end{aligned} \quad (29)$$

The last integral is:

$$\frac{(\alpha - \lambda)(\alpha - \lambda')}{a^2} \int_{-\infty}^{\infty} \delta(E - \lambda) \cdot \delta(E - \lambda') dE = \frac{(\alpha - \lambda)^2}{a^2} \delta(\lambda - \lambda') \quad (30)$$

Everything taken together:

$$\begin{aligned} & \int_{-\infty}^{\infty} f_{\lambda}^*(e) f_{\lambda'}(E) dE \\ & = c_{\lambda}^* c_{\lambda'} \int_{-\infty}^{\infty} [\rho a P \frac{1}{E - \lambda} + \frac{\alpha - \lambda}{a} \delta(E - \lambda)] \\ & \quad \cdot [\rho a P \frac{1}{E - \lambda'} + \frac{\alpha - \lambda'}{a} \delta(E - \lambda')] dE \\ & = c_{\lambda}^* c_{\lambda'} \{ [\frac{(\alpha - \lambda)^2}{a^2} + \rho^2 a^2 \pi^2] \delta(\lambda - \lambda') - \rho \} \end{aligned} \quad (31)$$

Thus we find:

$$\begin{aligned} \langle \tilde{\lambda} | \tilde{\lambda}' \rangle & = \frac{1}{\rho} \int_{-\infty}^{\infty} f_{\lambda}^*(E) f_{\lambda'}(E) dE + c_{\lambda}^* c_{\lambda'} \\ & = c_{\lambda}^* c_{\lambda'} [\frac{(\alpha - \lambda)^2}{\rho a^2} + \rho a^2 \pi^2] \delta(\lambda - \lambda') = \delta(\lambda - \lambda') \end{aligned} \quad (32)$$

and so we can make the following choice:

$$c_{\lambda} = \frac{-\sqrt{\rho} a}{\alpha - \lambda + i\pi \rho a^2} \quad (33)$$

so that:

$$f_\lambda(E) = \frac{-\sqrt{\rho}a}{\alpha - \lambda + i\pi\rho a^2} \left[\rho a P \frac{1}{E - \lambda} + \frac{\alpha - \lambda}{a} \delta(E - \lambda) \right] \quad (34)$$

Finally we find the following exact expression:

$$|\tilde{\lambda}\rangle = \frac{\sqrt{\rho}a}{\alpha - \lambda + i\pi\rho a^2} \left[\int_{-\infty}^{\infty} \rho a P \frac{1}{E - \lambda} |E\rangle dE + \frac{\alpha - \lambda}{a} |\lambda\rangle - |a\rangle \right] \quad (35)$$

The spectral representation of H becomes:

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \lambda |\tilde{\lambda}\rangle \langle \tilde{\lambda}| d\lambda \\ &= \int_{-\infty}^{\infty} \frac{\rho a^2 \lambda d\lambda}{(\alpha - \lambda)^2 + \pi^2 \rho^2 a^4} \left[\rho a \int_{-\infty}^{\infty} P \frac{1}{E - \lambda} |E\rangle dE + \frac{\alpha - \lambda}{a} |\lambda\rangle - |a\rangle \right] \\ &\quad \cdot \left[\rho a \int_{-\infty}^{\infty} P \frac{1}{E' - \lambda} \langle E'| dE' + \frac{\alpha - \lambda}{a} \langle \lambda| - \langle a| \right] \end{aligned} \quad (36)$$

This immediately leads to the following expression:

$$\begin{aligned} e^{-\frac{it}{\hbar}H} &= \int_{-\infty}^{\infty} \frac{\rho a^2 e^{-\frac{it}{\hbar}\lambda} d\lambda}{(\alpha - \lambda)^2 + \pi^2 \rho^2 a^4} \\ &\quad \cdot \left[\rho a \int_{-\infty}^{\infty} P \frac{1}{E - \lambda} |E\rangle dE + \frac{\alpha - \lambda}{a} |\lambda\rangle - |a\rangle \right] \\ &\quad \cdot \left[\rho a \int_{-\infty}^{\infty} P \frac{1}{E' - \lambda} \langle E'| dE' + \frac{\alpha - \lambda}{a} \langle \lambda| - \langle a| \right] \end{aligned} \quad (37)$$

From this it follows that:

$$\langle a | e^{-\frac{it}{\hbar}H} | a \rangle = \int_{-\infty}^{\infty} \frac{\rho a^2 e^{-\frac{it}{\hbar}\lambda}}{(\alpha - \lambda)^2 + \pi^2 \rho^2 a^4} d\lambda \quad (38)$$

This integral can be evaluated by contour integration. We have already assumed $t \geq 0$, therefore, by closing the contour with a semicircle at infinity in the lower half plane we find:

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\rho a^2 e^{-\frac{it}{\hbar}\lambda}}{(\alpha - \lambda)^2 + \pi^2 \rho^2 a^4} d\lambda \\ &= -2\pi i \lim_{\lambda \rightarrow \alpha - i\pi\rho a^2} \frac{\rho a^2 e^{-\frac{it}{\hbar}\lambda} (\lambda - \alpha + i\pi\rho a^2)}{(\lambda - \alpha)^2 + \pi^2 \rho^2 a^4} \\ &= -2\pi i \lim_{\lambda \rightarrow \alpha - i\pi\rho a^2} \frac{\rho a^2 e^{-\frac{it}{\hbar}\lambda}}{\lambda - \alpha - i\pi\rho a^2} \\ &= -2\pi i \frac{\rho a^2 e^{-\frac{it}{\hbar}(\alpha - i\pi\rho a^2)}}{-2i\pi\rho a^2} = e^{-\frac{it}{\hbar}(\alpha - i\pi\rho a^2)} \end{aligned} \quad (39)$$

Thus we find:

$$\langle a | e^{-\frac{it}{\hbar}H} | a \rangle = e^{-\frac{it}{\hbar}(\alpha - i\pi\rho a^2)} \quad (40)$$

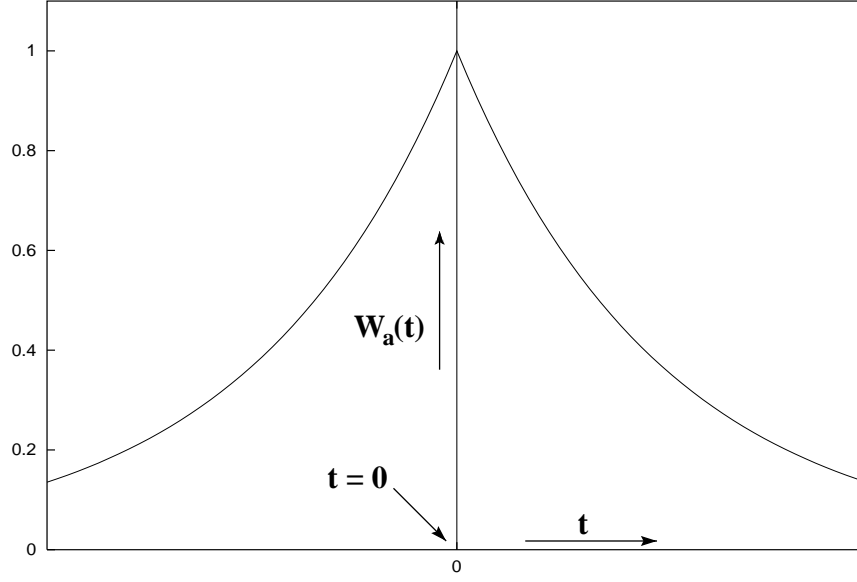


Figure 1: Formal survival rate as function of time.

In the course of time the probability of finding the original state back is equal to:

$$W_a(t) = |\langle a | e^{-\frac{it}{\hbar} H} | a \rangle|^2 = e^{-\frac{2\pi\rho a^2}{\hbar} t} \quad (41)$$

This is the exponential law of radioactive decay. Note that the coefficient of t in the exponent is in agreement with Fermi's Golden Rule.

Except for the simplified model specifications no approximation is made. Striking is that for *negative* t the same integral over λ generates a plus sign in the exponent, so that time symmetry is restored. There is apparently no question of time irreversibility. A simple kink in the time curve appears (see Figure 1).

3 Extension of the model

In Section 2 we assumed ρ and a to be independent of E . We now introduce E -dependence. We then have:

$$H_0|E\rangle = E|E\rangle, \quad H_0|a\rangle = \alpha|a\rangle, \quad -\infty < E < \infty \quad (42)$$

$$\langle E|E'\rangle = \frac{1}{\rho(E)}\delta(E - E'), \quad \langle E|a\rangle = \langle a|E\rangle = 0, \quad \langle a|a\rangle = 1 \quad (43)$$

$$\int_{-\infty}^{\infty} \rho(E)|E\rangle\langle E|dE + |a\rangle\langle a| = I, \quad \rho > 0 \quad (44)$$

and

$$H'|E\rangle = a(E)|a\rangle, \quad H'|a\rangle = \int_{-\infty}^{\infty} \rho(E)a(E)|E\rangle dE \quad (45)$$

Again we define:

$$|\tilde{\lambda}\rangle = \int_{-\infty}^{\infty} f_{\lambda}(E)|E\rangle dE + c_{\lambda}|a\rangle \quad (46)$$

and try to solve the eigenvalue equation:

$$(H - \lambda)|\tilde{\lambda}\rangle = (H_0 + H' - \lambda)|\tilde{\lambda}\rangle = 0 \quad (47)$$

with the normalization condition:

$$\langle \tilde{\lambda} | \tilde{\lambda}' \rangle = \int_{-\infty}^{\infty} \frac{1}{\rho(E)} f_{\lambda}^*(E) f_{\lambda'}(E) dE + c_{\lambda}^* c_{\lambda'} = \delta(\lambda - \lambda') \quad (48)$$

This now gives:

$$\begin{aligned} (H_0 + H' - \lambda)|\tilde{\lambda}\rangle &= (H_0 + H' - \lambda) \left[\int_{-\infty}^{\infty} f_{\lambda}(E)|E\rangle dE + c_{\lambda}|a\rangle \right] \\ &= \int_{-\infty}^{\infty} f_{\lambda}(E) [(E - \lambda)|E\rangle + a(E)|a\rangle] dE \\ &\quad + c_{\lambda} [(\alpha - \lambda)|a\rangle + \int_{-\infty}^{\infty} \rho(E)a(E)|E\rangle dE] = 0 \end{aligned} \quad (49)$$

This leads to:

$$(E - \lambda)f_{\lambda}(E) + \rho(E)a(E)c_{\lambda} = 0 \quad (50)$$

and

$$(\alpha - \lambda)c_{\lambda} + \int_{-\infty}^{\infty} a(E)f_{\lambda}(E)dE = 0 \quad (51)$$

So we find:

$$f_{\lambda}(E) = -\rho(E)a(E)c_{\lambda}P\frac{1}{E - \lambda} + \beta\delta(E - \lambda) \quad (52)$$

and

$$-c_{\lambda} \int_{-\infty}^{\infty} \rho(E)a^2(E)P\frac{1}{E - \lambda}dE + a(\lambda)\beta = -(\alpha - \lambda)c_{\lambda} \quad (53)$$

Now the integral is not zero. We find instead what is called the Hilbert transform³ of the function $\rho(E)a^2(E)$.

Define:

$$\eta(E) = \rho(E)a^2(E) \quad (54)$$

It is assumed that $\eta(E) \rightarrow 0$ sufficiently fast when $E \rightarrow \pm\infty$.

The Hilbert transform $\sigma(\lambda)$ of $\eta(E)$ is defined as:

$$\sigma(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} \eta(E)P\frac{1}{E - \lambda}dE \quad (55)$$

³Erdélyi, Bateman Manuscript Project, "Tables of Integral Transforms II", page 243. (McGraw Hill, 1954)

In terms of this function we then have:

$$\beta = -c_\lambda \frac{\alpha - \lambda - \pi\sigma(\lambda)}{a(\lambda)} \quad (56)$$

so we find:

$$f_\lambda(E) = -c_\lambda [\rho(E)a(E)P \frac{1}{E-\lambda} + \frac{\alpha - E - \pi\sigma(E)}{a(E)} \delta(E-\lambda)] \quad (57)$$

In order to make use of the normalization condition we evaluate:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{\rho(E)} f_\lambda^*(E) f_{\lambda'}(E) \\ = c_\lambda^* c_{\lambda'} \int_{-\infty}^{\infty} \frac{1}{\rho(E)} [\rho(E)a(E)P \frac{1}{E-\lambda} + \frac{\alpha - E - \pi\sigma(E)}{a(E)} \delta(E-\lambda)] \\ \cdot [\rho(E)a(E)P \frac{1}{E-\lambda'} + \frac{\alpha - E - \pi\sigma(E)}{a(E)} \delta(E-\lambda')] dE \end{aligned} \quad (58)$$

The integral:

$$\int_{-\infty}^{\infty} \rho(E)a^2(E)P \frac{1}{E-\lambda} P \frac{1}{E-\lambda'} dE = \int_{-\infty}^{\infty} \eta(E)P \frac{1}{E-\lambda} P \frac{1}{E-\lambda'} dE \quad (59)$$

can be determined and gives (see the Appendix):

$$\pi \frac{\sigma(\lambda) - \sigma(\lambda')}{\lambda - \lambda'} + \pi^2 \eta(\lambda) \delta(\lambda - \lambda') \quad (60)$$

The remaining terms give no problems and we end up with:

$$\begin{aligned} \langle \tilde{\lambda} | \tilde{\lambda}' \rangle &= \int_{-\infty}^{\infty} \frac{1}{\rho(E)} f_\lambda^*(E) f_{\lambda'}(E) dE + c_\lambda^* c_{\lambda'} \\ &= c_\lambda^* c_{\lambda'} \left\{ \frac{1}{\eta(\lambda)} [\alpha - \lambda - \pi\sigma(\lambda)]^2 + \eta(\lambda) \right\} \delta(\lambda - \lambda') = \delta(\lambda - \lambda') \end{aligned} \quad (61)$$

Apparently we can choose:

$$c_\lambda = \frac{-\sqrt{\eta(\lambda)}}{\alpha - \lambda - \pi\sigma(\lambda) + i\pi\eta(\lambda)} \quad (62)$$

This gives:

$$\begin{aligned} |\tilde{\lambda}\rangle &= \frac{\sqrt{\eta(\lambda)}}{\alpha - \lambda - \pi\sigma(\lambda) + i\pi\eta(\lambda)} \left[\int_{-\infty}^{\infty} \rho(E)a(E)P \frac{1}{E-\lambda} |E\rangle dE \right. \\ &\quad \left. + \frac{\alpha - \lambda - \pi\sigma(\lambda)}{a(\lambda)} |\lambda\rangle - |a\rangle \right] \end{aligned} \quad (63)$$

From this it follows that:

$$\langle a|\tilde{\lambda}\rangle = \frac{-\sqrt{\eta(\lambda)}}{\alpha - \lambda - \pi\sigma(\lambda) + i\pi\eta(\lambda)} \quad (64)$$

and thus we obtain:

$$\begin{aligned} \langle a|e^{-\frac{it}{\hbar}H}|a\rangle &= \int_{-\infty}^{\infty} e^{-\frac{it}{\hbar}\lambda} \langle a|\tilde{\lambda}\rangle \langle \tilde{\lambda}|a\rangle d\lambda \\ &= \int_{-\infty}^{\infty} \frac{\eta(\lambda)e^{-\frac{it}{\hbar}\lambda} d\lambda}{[\alpha - \lambda - \pi\sigma(\lambda)]^2 + \pi^2\eta^2(\lambda)} \end{aligned} \quad (65)$$

This is again an exact expression. If a is small also σ is small and a good approximation can be obtained by writing:

$$\begin{aligned} \langle a|e^{-\frac{it}{\hbar}H}|a\rangle &\approx \int_{-\infty}^{\infty} \frac{\eta(\alpha)e^{-\frac{it}{\hbar}\lambda} d\lambda}{[\alpha - \lambda - \pi\sigma(\alpha)]^2 + \pi^2\eta^2(\alpha)} \\ &= e^{-\frac{it}{\hbar}[\alpha - \pi\sigma(\alpha) - i\pi\eta(\alpha)]} \end{aligned} \quad (66)$$

Note that this is again in agreement with Fermi's Golden Rule.

Let us now consider the exact integral expression. By making use of theorems on Hilbert transforms we can get insight into the analytical properties of the integrand. The relevant theorems are proven in the Appendix and sound:

1. If $\sigma(y)$ is the Hilbert transform of $\eta(x)$:

$$\sigma(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \eta(x) P \frac{1}{x-y} dx \quad (67)$$

then $-\eta(y)$ is the Hilbert transform of $\sigma(x)$:

$$\eta(y) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \sigma(x) P \frac{1}{x-y} dx \quad (68)$$

2. The function:

$$\xi(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\sigma(x) + i\eta(x)}{x-z} dx \quad (69)$$

is zero in the lower half z -plane, analytic and regular in the upper half z -plane and has the property:

$$\lim_{z \downarrow x_0} \xi(z) = \sigma(x_0) + i\eta(x_0) \quad (70)$$

Here x and x_0 are points on the real axis. The function can be analytically continued from the upper to the lower half z -plane but may not be regular there. It is clear that $\xi^*(z)$ has just the opposite properties.

3. The function:

$$\tilde{\sigma}(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \eta(z) P \frac{1}{z-x} P \frac{1}{z-y} dz \quad (71)$$

satisfies the property:

$$\tilde{\sigma}(x, y) = \frac{\sigma(x) - \sigma(y)}{x - y} + \pi\eta(x)\delta(x - y) \quad (72)$$

The latter theorem has been used before and will be used later on. We can now rewrite the matrix element in terms of ξ , ξ^* and η :

$$\langle a | e^{-\frac{it}{\hbar}H} | a \rangle = \int_{-\infty}^{\infty} \frac{\eta(\lambda) e^{-\frac{it}{\hbar}\lambda} d\lambda}{[\alpha - \lambda - \pi\xi(\lambda)][\alpha - \lambda - \pi\xi^*(\lambda)]} \quad (73)$$

If $\eta(\lambda)$ is regular in the lower half λ -plane except for poles then since:

$$\eta(\lambda) = \frac{\xi(\lambda) - \xi^*(\lambda)}{2i} \quad (74)$$

and $\xi^*(\lambda)$ is regular in the lower half plane, also (the analytical continuation of) $\xi(\lambda)$ is regular in the lower half plane except for poles. For $t \neq 0$ we can split the integral into two parts:

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{\eta(\lambda) e^{-\frac{it}{\hbar}\lambda} d\lambda}{[\alpha - \lambda - \pi\xi(\lambda)][\alpha - \lambda - \pi\xi^*(\lambda)]} \\ &= \frac{1}{2\pi i} \left[\int_{-\infty}^{\infty} \frac{e^{-\frac{it}{\hbar}\lambda} d\lambda}{\alpha - \lambda - \pi\xi(\lambda)} - \int_{-\infty}^{\infty} \frac{e^{-\frac{it}{\hbar}\lambda} d\lambda}{\alpha - \lambda - \pi\xi^*(\lambda)} \right] \end{aligned} \quad (75)$$

In order to evaluate this expression we have to solve the equation:

$$\alpha - \lambda - \pi\xi(\lambda) = 0 \quad (76)$$

It is interesting to see what happens when we introduce a scaling factor and replace η by $\tau\eta$ ($\tau > 0$). Then ξ is replaced by $\tau\xi$. We then have to solve the equation:

$$\alpha - \lambda - \pi\tau\xi(\lambda) = 0 \quad (77)$$

When τ approaches zero one of the roots approaches α . Then in first approximation we have:

$$\alpha - \lambda - \pi\tau\xi(\alpha) = 0 \quad (78)$$

and we find:

$$\lambda_0 \approx \alpha - \pi\tau\xi(\alpha) = \alpha - \pi\tau[\sigma(\alpha) + i\eta(\alpha)] \quad (79)$$

This is what we have seen before. Since $\eta(\alpha) > 0$ this zero lies in the lower half plane and since this root never becomes real for any value of τ it stays in the lower half plane for all values of τ .

The other roots approach poles in $\xi(\lambda)$. Let λ' be such a pole. Then, near λ' , we have:

$$\xi(\lambda) \approx \frac{r_{\lambda'}}{\lambda - \lambda'} \quad (80)$$

and we find:

$$\lambda_0 \approx \lambda' + \pi\tau \frac{r_{\lambda'}}{\alpha - \lambda'} \quad (81)$$

Since $\xi(\lambda)$ is regular in the upper half plane, for sufficiently small τ these roots are all located in the lower half plane and since the roots are never real for any value of τ they remain in the lower half plane for all values of τ . Meanwhile, $\xi^*(\lambda)$ is regular in the lower half plane, so for sufficiently small τ there cannot be solutions of the equation:

$$\alpha - \lambda - \pi\tau\xi^*(\lambda) = 0 \quad (82)$$

located in the lower half plane. Now when we let τ move from infinitesimal to regular values zero's will not move in and cannot spontaneously be created so for any τ there will be no solutions of the equation in the lower half plane. For $t > 0$ the conclusion is that:

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{it}{\hbar}\lambda} d\lambda}{\alpha - \lambda - \pi\xi^*(\lambda)} = 0 \quad (83)$$

and we find:

$$\langle a | e^{-\frac{it}{\hbar}H} | a \rangle = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-\frac{it}{\hbar}\lambda} d\lambda}{\alpha - \lambda - \pi\xi(\lambda)} \quad (84)$$

Let λ_0 be a root. Then $\Im\lambda_0 < 0$ and its contribution to the integral is:

$$-\frac{1}{\pi\xi'(\lambda_0) + 1} e^{-\frac{it}{\hbar}\lambda_0} \quad (85)$$

We end up with the following expression for $t \geq 0$:

$$\langle a | e^{-\frac{it}{\hbar}H} | a \rangle = \sum_{\lambda_0} \gamma_{\lambda_0} e^{-\frac{it}{\hbar}\Re\lambda_0} e^{-\frac{t}{\hbar}|\Im\lambda_0|} \quad (86)$$

where the γ satisfy the necessary but not sufficient condition:

$$\sum_{\lambda_0} \gamma_{\lambda_0} = 1 \quad (87)$$

The result is a sum of exponentials decreasing with time. One of them ultimately will dominate. Note that the time derivative of

$$W_a(t) = |\langle a | e^{-\frac{it}{\hbar}H} | a \rangle|^2 \quad (88)$$

in the limit $t \downarrow 0$ is not zero. Still there is time reversal symmetry. Therefore again a kink appears at $t = 0$ and the time derivative at $t = 0$ does not exist.

4 The influence of the energy lower bound

Without loss of generality we may assume that $E = 0$ is the lowest energy value of the free Hamiltonian H_0 . In that case we have, with α assumed $\neq 0$:

$$H_0|E\rangle = E|E\rangle, \quad H_0|a\rangle = \alpha|a\rangle, \quad 0 < E < \infty \quad (89)$$

$$\langle E|E'\rangle = \frac{1}{\rho(E)}\delta(E-E'), \quad \langle E|a\rangle = \langle a|E\rangle = 0, \quad \langle a|a\rangle = 1 \quad (90)$$

$$\int_0^\infty \rho(E)|E\rangle\langle E|dE + |a\rangle\langle a| = I, \quad \rho > 0 \quad (91)$$

and

$$H'|E\rangle = a(E)|a\rangle, \quad H'|a\rangle = \int_0^\infty \rho(E)a(E)|E\rangle dE \quad (92)$$

As before we consider normalized eigenkets of the Hamiltonian $H = H_0 + H'$. We put H in its spectral form:

$$H = \int_{-\infty}^\infty \lambda|\tilde{\lambda}\rangle\langle\tilde{\lambda}|d\lambda \quad (93)$$

$$H|\tilde{\lambda}\rangle = \lambda|\tilde{\lambda}\rangle \quad (94)$$

$$\langle\tilde{\lambda}|\tilde{\lambda}'\rangle = \delta(\lambda - \lambda') \quad (95)$$

$$\int_{-\infty}^\infty |\tilde{\lambda}\rangle\langle\tilde{\lambda}|d\lambda = I \quad (96)$$

Note here that we do not assume a lower bound on the eigenvalues. We now write:

$$|\tilde{\lambda}\rangle = \int_0^\infty f_\lambda(E)|E\rangle dE + c_\lambda|a\rangle \quad (97)$$

The eigenvalue equation gives:

$$\begin{aligned} (H_0 + H' - \lambda)|\tilde{\lambda}\rangle &= (H_0 + H' - \lambda)\left[\int_0^\infty f_\lambda(E)|E\rangle dE + c_\lambda|a\rangle\right] \\ &= \int_0^\infty f_\lambda(E)[(E - \lambda)|E\rangle + a(E)|a\rangle]dE \\ &\quad + c_\lambda[(\alpha - \lambda)|a\rangle + \int_0^\infty \rho(E)a(E)|E\rangle dE] = 0 \end{aligned} \quad (98)$$

from which it follows that:

$$(E - \lambda)f_\lambda(E) + \rho(E)a(E)c_\lambda = 0 \quad (99)$$

and

$$(\alpha - \lambda)c_\lambda + \int_0^\infty a(E)f_\lambda(E)dE = 0 \quad (100)$$

This gives, as before:

$$f_\lambda(E) = -\rho(E)a(E)c_\lambda P \frac{1}{E - \lambda} + \beta\delta(E - \lambda) \quad (101)$$

When $\lambda > 0$ we obtain an equation for β :

$$-c_\lambda \int_0^\infty \eta(E)P \frac{1}{E - \lambda} dE + a(\lambda)\beta = -(\alpha - \lambda)c_\lambda \quad (102)$$

which can be solved:

$$\beta = c_\lambda \frac{\int_0^\infty \eta(E) P \frac{1}{E-\lambda} dE - (\alpha - \lambda)}{a(\lambda)} \quad (103)$$

In that case we have:

$$f_\lambda(E) = -c_\lambda [\rho(E) a(E) P \frac{1}{E-\lambda} + \frac{\alpha - E - \pi \bar{\sigma}(E)}{a(E)} \delta(E - \lambda)] \quad (104)$$

where:

$$\bar{\sigma}(\lambda) = \frac{1}{\pi} \int_0^\infty \eta(E) P \frac{1}{E-\lambda} dE = \frac{1}{\pi} \int_{-\infty}^\infty \eta(E) \theta(E) P \frac{1}{E-\lambda} dE \quad (105)$$

and is therefore the Hilbert transform of the function $\eta(E)\theta(E)$. We define:

$$\bar{\eta}(E) = \eta(E)\theta(E) \quad (106)$$

We get the following expression for the eigenkets of H:

$$\begin{aligned} |\tilde{\lambda}\rangle = & -c_\lambda \int_0^\infty \frac{1}{a(E)} \{ \eta(E) P \frac{1}{E-\lambda} \\ & + [\alpha - E - \pi \bar{\sigma}(E)] \delta(E - \lambda) \} |E\rangle dE + c_\lambda |a\rangle \end{aligned} \quad (107)$$

We evaluate:

$$\begin{aligned} & \int_0^\infty \frac{1}{\rho(E)} f_\lambda^*(E) f_{\lambda'}(E) dE \\ & = c_\lambda^* c_{\lambda'} \int_0^\infty \frac{1}{\eta(E)} \{ \eta(E) P \frac{1}{E-\lambda} + [\alpha - E - \pi \bar{\sigma}(E)] \delta(E - \lambda) \} \\ & \quad \cdot \{ \eta(E) P \frac{1}{E-\lambda'} + [\alpha - E - \pi \bar{\sigma}(E)] \delta(E - \lambda') \} dE \end{aligned} \quad (108)$$

Here we have:

$$\begin{aligned} & \int_0^\infty \eta(E) P \frac{1}{E-\lambda} P \frac{1}{E-\lambda'} dE = \int_{-\infty}^\infty \bar{\eta}(E) P \frac{1}{E-\lambda} P \frac{1}{E-\lambda'} dE \\ & = \pi \frac{\bar{\sigma}(\lambda) - \bar{\sigma}(\lambda')}{\lambda - \lambda'} + \pi^2 \bar{\eta}(\lambda) \delta(\lambda - \lambda') \end{aligned} \quad (109)$$

Again the remaining terms can immediately be evaluated and the result becomes:

$$\begin{aligned} \langle \tilde{\lambda} | \tilde{\lambda}' \rangle & = \int_0^\infty \frac{1}{\rho(E)} f_\lambda^*(E) f_{\lambda'}(E) dE + c_\lambda^* c_{\lambda'} \\ & = c_\lambda^* c_{\lambda'} \{ \frac{1}{\eta(\lambda)} [\alpha - \lambda - \pi \bar{\sigma}(\lambda)]^2 + \bar{\eta}(\lambda) \} \delta(\lambda - \lambda') = \delta(\lambda - \lambda') \end{aligned} \quad (110)$$

for $\lambda, \lambda' > 0$. This allows the normalization constant to be determined except for a phase. We choose:

$$c_\lambda = \frac{-\sqrt{\bar{\eta}(\lambda)}}{\alpha - \lambda - \pi\bar{\sigma}(\lambda) + i\pi\bar{\eta}(\lambda)} = \frac{-\sqrt{\bar{\eta}(\lambda)}}{\alpha - \lambda - \pi\bar{\xi}^*(\lambda)} \quad (111)$$

Note the difference with the former expression. Moreover it is only valid for $\lambda > 0$.

Finally we obtain for positive λ :

$$\begin{aligned} |\tilde{\lambda}\rangle = & \frac{\sqrt{\bar{\eta}(\lambda)}}{\alpha - \lambda - \pi\bar{\xi}^*(\lambda)} \left[\int_0^\infty \frac{1}{a(E)} \eta(E) P \frac{1}{E - \lambda} |E\rangle dE \right. \\ & \left. + \frac{\alpha - \lambda - \pi\bar{\sigma}(\lambda)}{a(\lambda)} |\lambda\rangle - |a\rangle \right] \end{aligned} \quad (112)$$

and so we get:

$$\langle a | \tilde{\lambda} \rangle = \frac{-\sqrt{\bar{\eta}(\lambda)}}{\alpha - \lambda - \pi\bar{\xi}^*(\lambda)} \quad (113)$$

When $\lambda < 0$ it is not possible to solve for β because then $\delta(E - \lambda)$ is always zero. In that case we have:

$$f_\lambda(E) = -\frac{\rho(E)a(E)}{E - \lambda} c_\lambda \quad (114)$$

When this is inserted we obtain:

$$\int_0^\infty \frac{\rho(E)a^2(E)}{E - \lambda} dE = \int_0^\infty \frac{\eta(E)}{E - \lambda} dE = \alpha - \lambda \quad (115)$$

which gives:

$$\alpha - \lambda - \pi\bar{\sigma}(\lambda) = 0 \quad (116)$$

This is an equation for λ to be solved. We have for $\lambda < 0$ that the integral and all its derivatives are positive so for any α with the property:

$$\alpha < \lim_{\lambda \uparrow 0} \bar{\sigma}(\lambda) \quad (117)$$

there is one and only one solution. If α is larger then there is no solution. It might be that this limit is positive infinite. Then there is always one solution. The solution is to be interpreted as a bound state. Let λ_0 be this solution. We have for $\lambda < 0$:

$$|\tilde{\lambda}\rangle = -c_\lambda \int_0^\infty \frac{1}{a(E)} \frac{\eta(E)}{E - \lambda} |E\rangle dE + c_\lambda |a\rangle \quad (118)$$

This gives:

$$\begin{aligned}
\langle \tilde{\lambda} | \tilde{\lambda}' \rangle &= c_{\lambda}^* c_{\lambda'} \left[\int_0^{\infty} \frac{\eta(E) dE}{(E - \lambda)(E - \lambda')} + 1 \right] \\
&= c_{\lambda}^* c_{\lambda'} \left[\int_{-\infty}^{\infty} \bar{\eta}(E) P \frac{1}{E - \lambda} P \frac{1}{E - \lambda'} dE + 1 \right] \\
&= c_{\lambda}^* c_{\lambda'} \left[\pi \frac{\bar{\sigma}(\lambda) - \bar{\sigma}(\lambda')}{\lambda - \lambda'} + \pi^2 \bar{\eta}(\lambda) \delta(\lambda - \lambda') + 1 \right] \\
&= c_{\lambda}^* c_{\lambda'} \left[\pi \frac{\bar{\sigma}(\lambda) - \bar{\sigma}(\lambda')}{\lambda - \lambda'} + 1 \right]
\end{aligned} \tag{119}$$

From this it follows that:

$$\langle \tilde{\lambda}_0 | \tilde{\lambda}_0 \rangle = c_{\lambda_0}^* c_{\lambda_0} [\pi \bar{\sigma}'(\lambda_0) + 1] = 1 \tag{120}$$

and thus we have:

$$c_{\lambda_0}^* c_{\lambda_0} = \frac{1}{\pi \bar{\sigma}'(\lambda_0) + 1} \tag{121}$$

where:

$$\bar{\sigma}'(\lambda_0) = \frac{1}{\pi} \int_0^{\infty} \frac{\eta(E)}{(E - \lambda_0)^2} dE > 0 \tag{122}$$

The following expression results if there is a bound state:

$$\begin{aligned}
\langle a | e^{-\frac{i\epsilon}{\hbar} \lambda} | a \rangle &= \int_0^{\infty} e^{-\frac{i\epsilon}{\hbar} \lambda} \langle a | \tilde{\lambda} \rangle \langle \tilde{\lambda} | a \rangle d\lambda + e^{-\frac{i\epsilon}{\hbar} \lambda_0} \langle a | \tilde{\lambda}_0 \rangle \langle \tilde{\lambda}_0 | a \rangle \\
&= \int_0^{\infty} \frac{\bar{\eta}(\lambda) e^{-\frac{i\epsilon}{\hbar} \lambda} d\lambda}{[\alpha - \lambda - \pi \bar{\sigma}(\lambda)][\alpha - \lambda - \pi \bar{\sigma}^*(\lambda)]} + \frac{e^{-\frac{i\epsilon}{\hbar} \lambda_0}}{\pi \bar{\sigma}'(\lambda_0) + 1}
\end{aligned} \tag{123}$$

The second term on the right hand side oscillates forever. That means that when one prepares the state $|a\rangle$ this state may contain a contribution from a bound state which does not decay. The first term can be rewritten in the form:

$$\frac{1}{2\pi i} \left[\int_0^{\infty} \frac{e^{-\frac{i\epsilon}{\hbar} \lambda} d\lambda}{\alpha - \lambda - \pi \bar{\xi}(\lambda)} - \int_0^{\infty} \frac{e^{-\frac{i\epsilon}{\hbar} \lambda} d\lambda}{\alpha - \lambda - \pi \bar{\xi}^*(\lambda)} \right] \tag{124}$$

Since the integrands have a possible pole on the negative real axis, extension of the integration path to minus infinity is not immediately possible. We have to avoid this pole and we do that by choosing the path of integration for the first integral as shown in Figure 2.

If the same integration path were chosen for the second integral the extension of the integration path to minus infinity would not have changed the results, because the integrands are equal. However, then the second integral would not be zero. This integral is only zero when the integration path be chosen as in Figure 3.

The difference is just the contribution from the bound state pole and thus we find that:

$$\langle a | e^{-\frac{i\epsilon}{\hbar} H} | a \rangle = \frac{1}{2\pi i} \int \frac{e^{-\frac{i\epsilon}{\hbar} \lambda} d\lambda}{\alpha - \lambda - \pi \bar{\xi}(\lambda)} \tag{125}$$

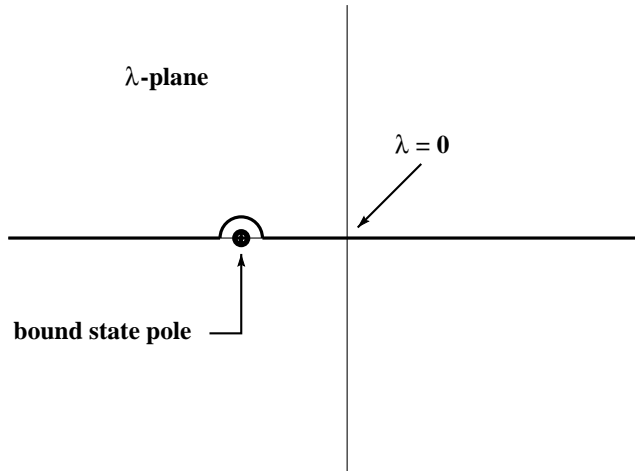


Figure 2: Integration path of the first integral.

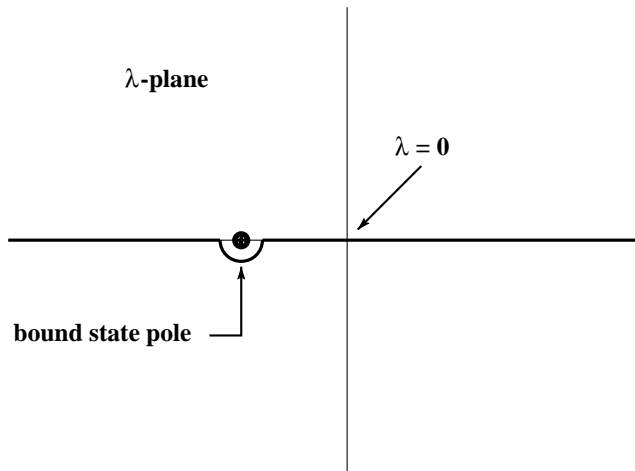


Figure 3: Integration path of the second integral.

with the path of integration taken along the entire real axis except for the bound state pole and which is passed through the upper half plane, i.e. the contour is that of Figure 2. The integrand is analytic and regular in the upper half plane (on this particular Riemann sheet) except for the pole on the negative real axis. The right hand side of this expression causes trouble because the functions $\bar{\eta}(\lambda)$ and $\bar{\sigma}(\lambda)$ are definitely nonanalytic in the lower half plane and a contour integration in the way used above is not possible. Suppose that $\eta(\lambda)$ be analytic and regular except for poles; $\eta(\lambda) > 0$ and finite for $\lambda > 0$; $\eta(0)$ finite. Then $\eta(\lambda)\theta(\lambda)$ is "analytic" in the sense that it has a branch cut which on both sides of the real axis stretches itself out towards infinity and passes through the origin. The cut separates two analytic functions, one of them being identically zero, the other being $\eta(\lambda)$. By using a similar argument as before one can now easily prove that $\bar{\xi}(\lambda)$ is still regular in the upper half plane and that its analytic continuation is regular in the lower half plane except for poles and a branch cut starting at the origin. Now the point $\lambda = 0$ is necessarily a true branch point. We take the cut along the negative imaginary axis. Except for the bound state pole all poles are on the right hand side of the cut and in the lower half plane. In order to evaluate the integral we deform the contour as to wrap the cut: The right hand side is rotated clockwise and the left hand side anticlockwise towards the negative imaginary axis. In the mean time poles are passed which give exponential contributions to the integral. The problem now is to evaluate the remaining contour integral. In Figure 4 the branch cut, possible poles and the original integration path are drawn. Note how the branch point is evaded.

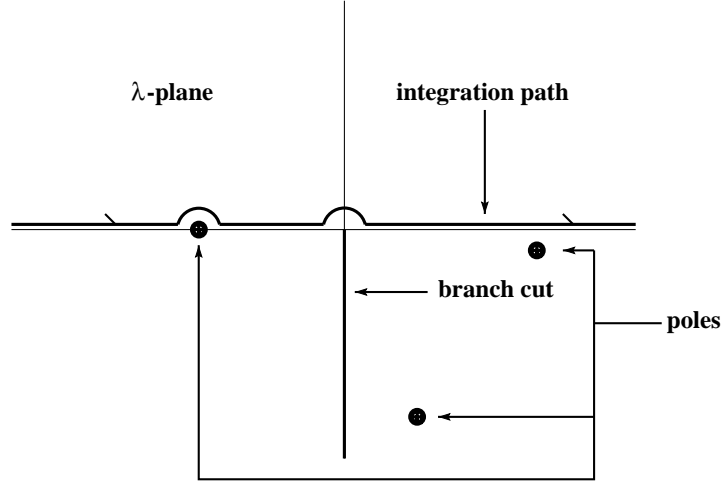


Figure 4: Intermediate integration path.

After deformation of the contour the situation is as sketched in Figure 5. Let Γ be the final contour. The poles on the right hand side of the imaginary axis give damped oscillatory contributions, the pole on the left hand side gives a pure oscillatory contribution from the bound state.

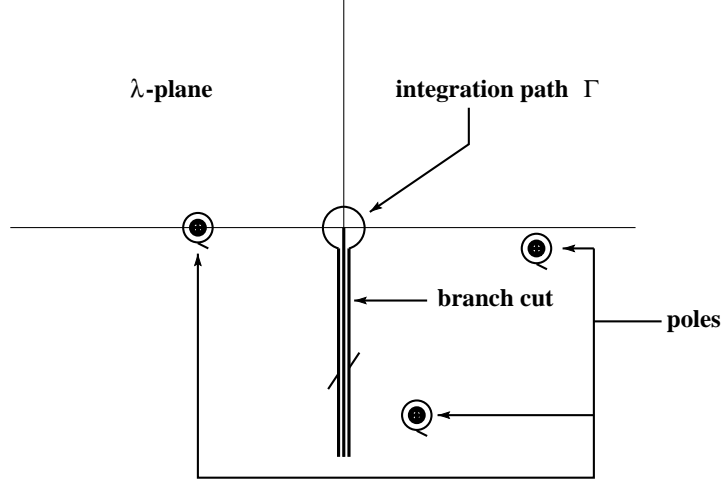


Figure 5: Final integration path.

The remaining contour integral is:

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{-\frac{i}{\hbar}\lambda} d\lambda}{\alpha - \lambda - \pi \bar{\xi}(\lambda)} &= \frac{1}{2\pi i} \int_0^{-i\infty} F(\lambda) e^{-\frac{i}{\hbar}\lambda} d\lambda \\ &= -\frac{1}{2\pi} \int_0^{\infty} F(-i\lambda) e^{-\frac{i}{\hbar}\lambda} d\lambda \end{aligned} \quad (126)$$

where $F(\lambda)$ is the "jump across the cut" of the function:

$$\frac{1}{\alpha - \lambda - \pi \bar{\xi}(\lambda)} \quad (127)$$

The integral is apparently a linear combination of an infinite number of decaying exponentials. The function $F(\lambda)$ cannot be identically equal to zero between 0 and some point on the negative imaginary axis, otherwise $\lambda = 0$ is not a true branch point. We find therefore that for $t \rightarrow \infty$ the integral will dominate any exponential of the type $e^{-\frac{t}{\tau}}$ for $\tau > 0$. The conclusion is therefore that there will be a non-exponential contribution to the transition matrix element which will ultimately dominate any exponential decay. This is the anomaly.

Let us now discuss some details of this anomaly. Let $\eta(\lambda)$ be a real, rational analytic function of λ , positive when λ is positive. Then for *negative* λ we find that:

$$\bar{\sigma}(\lambda) = -\frac{1}{\pi} \eta(\lambda) \ln\left(-\frac{\lambda}{c}\right) + \frac{1}{\pi} A(\lambda, c) \quad (128)$$

with c a positive number with the dimension of λ . That branch is chosen where the logarithm is real. Then the function $A(\lambda, c)$ is a real and rational analytic function of λ , finite around $\lambda = 0$ and which for negative λ is sufficiently positive. Now we consider two cases:

1. $\eta(0) > 0$. Then for small λ :

$$\alpha - \lambda - \pi \bar{\xi}(\lambda) = \alpha - \lambda - \pi \bar{\sigma}(\lambda) \approx \eta(\lambda) \ln\left(-\frac{\lambda}{c}\right) \quad (129)$$

The "jump across the cut" of the integrand is then:

$$F(\lambda) \rightarrow \frac{1}{\eta(\lambda) \ln\left(-\frac{\lambda}{c}\right)} - \frac{1}{\eta(\lambda) [\ln\left(-\frac{\lambda}{c}\right) + 2\pi i]} \rightarrow 0 \quad (\lambda \rightarrow 0) \quad (130)$$

2. $\eta(0) = 0$. We have:

$$\begin{aligned} \alpha - \lambda - \pi \bar{\xi}(\lambda) &= \alpha - \lambda - \pi \bar{\sigma}(\lambda) \\ &= \alpha - \lambda + \eta(\lambda) \ln\left(-\frac{\lambda}{c}\right) - A(\lambda, c) \end{aligned} \quad (131)$$

and we find then for small λ :

$$\begin{aligned} F(\lambda) &= \frac{1}{\alpha - \lambda + \eta(\lambda) \ln\left(-\frac{\lambda}{c}\right) - A(\lambda, c)} \\ &\quad - \frac{1}{\alpha - \lambda + \eta(\lambda) [\ln\left(-\frac{\lambda}{c}\right) + 2\pi i] - A(\lambda, c)} \\ &\approx \frac{2\pi i \eta(\lambda)}{[\alpha - \lambda - A(\lambda, c)]^2} \rightarrow 0 \quad (\lambda \rightarrow 0) \end{aligned} \quad (132)$$

The conclusion is that the transition matrix element behaves non-exponentially but goes faster to zero than t^{-1} for large t . Correspondingly the survival rate goes slower than exponential but faster than t^{-2} to zero for large t . Of course, the anomaly has been demonstrated only in a simplified model which however is realistic enough to warrant the expectation that the anomaly is characteristic for a much wider class of models. It may even be unavoidable.

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A Appendix

In this Appendix we prove the three theorems on Hilbert transforms mentioned in the text.

1. Let $\sigma(y)$ be the Hilbert transform of $\eta(x)$:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \eta(x) P \frac{1}{x-y} dx = \sigma(y) \quad (133)$$

then $-\eta(y)$ is the Hilbert transform of $\sigma(x)$:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \sigma(x) P \frac{1}{x-y} dx = -\eta(y) \quad (134)$$

Proof: In the text we have already proven that

$$\frac{1}{\pi^2} \int_{-\infty}^{\infty} P \frac{1}{x-z} P \frac{1}{y-z} dz = \delta(x-y) \quad (135)$$

We now have:

$$\begin{aligned} & \int_{-\infty}^{\infty} \eta(x) \delta(x-y) dx \\ &= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \eta(x) P \frac{1}{x-z} dx \right] P \frac{1}{y-z} dz = \eta(y) \end{aligned} \quad (136)$$

and therefore:

$$\eta(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \sigma(z) P \frac{1}{y-z} dz = -\frac{1}{\pi} \int_{-\infty}^{\infty} \sigma(z) P \frac{1}{z-y} dz \quad (137)$$

2. The function:

$$\xi(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\sigma(x) + i\eta(x)}{x-z} dx \quad (138)$$

is zero in the lower half z -plane, analytic and regular in the upper half z -plane and has the property:

$$\lim_{z \downarrow x_0} \xi(z) = \sigma(x_0) + i\eta(x_0) \quad (139)$$

Here x and x_0 are points on the real axis.

Proof: The function is obviously analytic and regular in the upper and lower half z -plane. Since the real axis acts as a closed branch cut the two

branches are not analytically connected. We have:

$$\begin{aligned}
\lim_{z \downarrow x_0} \xi(z) &= \frac{1}{2\pi i} \lim_{z \downarrow x_0} \int_{-\infty}^{\infty} \frac{\sigma(x) + i\eta(x)}{x - z} dx \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\sigma(x) + i\eta(x)}{x - x_0 - i\epsilon} dx \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} [\sigma(x) + i\eta(x)] \cdot [P \frac{1}{x - x_0} + \pi i \delta(x - x_0)] dx \\
&= \frac{1}{2i} [-\eta(x_0) + i\sigma(x_0)] + \frac{1}{2} [\sigma(x_0) + i\eta(x_0)] \\
&= \sigma(x_0) + i\eta(x_0)
\end{aligned} \tag{140}$$

This proves the first part. We also have:

$$\begin{aligned}
\lim_{z \uparrow x_0} \xi(z) &= \frac{1}{2\pi i} \lim_{z \uparrow x_0} \int_{-\infty}^{\infty} \frac{\sigma(x) + i\eta(x)}{x - z} dx \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\sigma(x) + i\eta(x)}{x - x_0 + i\epsilon} dx \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} [\sigma(x) + i\eta(x)] \cdot [P \frac{1}{x - x_0} - \pi i \delta(x - x_0)] dx \\
&= \frac{1}{2i} [-\eta(x_0) + i\sigma(x_0)] - \frac{1}{2} [\sigma(x_0) + i\eta(x_0)] = 0
\end{aligned} \tag{141}$$

This means that the regular function in the lower half plane is zero on the real axis and must therefore be zero throughout the lower half plane.

3. The function:

$$\tilde{\sigma}(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \eta(z) P \frac{1}{z - x} P \frac{1}{z - y} dz \tag{142}$$

satisfies the property:

$$\tilde{\sigma}(x, y) = \frac{\sigma(x) - \sigma(y)}{x - y} + \pi \eta(x) \delta(x - y) \tag{143}$$

Proof:

$$\begin{aligned}
&\frac{1}{\pi} \int_{-\infty}^{\infty} z \eta(z) P \frac{1}{z - x} P \frac{1}{z - y} dz \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} [(z - x) + x] \eta(z) P \frac{1}{z - x} P \frac{1}{z - y} dz \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \eta(z) P \frac{1}{z - y} dz + x \frac{1}{\pi} \int_{-\infty}^{\infty} \eta(z) P \frac{1}{z - x} P \frac{1}{z - y} dz \\
&= \sigma(y) + x \tilde{\sigma}(x, y) = \sigma(x) + y \tilde{\sigma}(x, y)
\end{aligned} \tag{144}$$

It follows from this that:

$$(x - y)\tilde{\sigma}(x, y) = \sigma(x) - \sigma(y) \quad (145)$$

and from this we obtain:

$$\tilde{\sigma}(x, y) = \frac{\sigma(x) - \sigma(y)}{x - y} + A(x)\delta(x - y) \quad (146)$$

The $A(x)$ can be evaluated:

$$\begin{aligned} A(x) &= \int_{-\infty}^{\infty} \tilde{\sigma}(x, y)dy - \int_{-\infty}^{\infty} \frac{\sigma(x) - \sigma(y)}{x - y}dy \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \eta(z)P\frac{1}{z - x} \left[\int_{-\infty}^{\infty} P\frac{1}{z - y}dy \right] dz \\ &\quad - \sigma(x) \int_{-\infty}^{\infty} P\frac{1}{x - y}dy + \int_{-\infty}^{\infty} \sigma(y)P\frac{1}{x - y}dy \\ &= \pi\eta(x) \end{aligned} \quad (147)$$